UE Programming for data analysis 2022 Exercise set Lin.Alg.

Exercises for Section 1: Linear Algebra

In the following, all products between matrixes are noted AB (for the product of matrix A and B). These are matrix multiplication as seen during the course.

1. Represent the following vectors of $\mathbb{R}^2$on a plane with two orthogonal axes. Vectors are written in *matrix* notation.

   (a) $\begin{bmatrix}0 & 2\end{bmatrix}$
   (b) $\begin{bmatrix}3 & -1\end{bmatrix}$
   (c) $2\begin{bmatrix}1 & -3\end{bmatrix}$
   (d) $\begin{bmatrix}2 & 4\end{bmatrix} - \begin{bmatrix}0 & -2\end{bmatrix}$
   (e) $\begin{bmatrix}1 & 4\end{bmatrix} + 3\begin{bmatrix}2 & -1\end{bmatrix}$
   (f) $5\begin{bmatrix}1 & 1\end{bmatrix} + 2\begin{bmatrix}-2 & 1\end{bmatrix}$
   (g) $\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}$
   (h) $\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}$
   (i) $\begin{bmatrix}0 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix}$
   (j) $\left(\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix} + \begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{bmatrix}1\\ 3\end{bmatrix}$

2. Find the scalar k for which the vector $k\begin{bmatrix}1 & 2 & 1 & 3 & 1\end{bmatrix}$ is a unit vector of $\mathbb{R}^5$ (i.e. the norm of the vector is one).

3. Provide M and P to write the following equation system in the form $M\vec{v} = P$, where vector $\vec{v} = \begin{bmatrix}x & y & z\end{bmatrix}^T$ belongs to $\mathbb{R}^3$. Solve the following two equation systems.

$$\begin{aligned} x + y + 2z &= 3 \\ x + 2y + z &= 1 \\ 2x + y + z &= 0 \end{aligned} \qquad \begin{aligned} x + 2z &= 1 \\ -y + z &= 2 \\ x - 2y &= 1 \end{aligned}$$

4. First write out what is the dimensionality of the products shown here, then calculate the products.

$$x^T y = \begin{bmatrix}1 - 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \quad Wx = \begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad XX^T = \begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\end{bmatrix}$$

5. Find all possible products among the following matrices:

$$A = \begin{bmatrix}1 & 2 & 3\end{bmatrix}, \qquad B = \begin{bmatrix}1\\ -2\end{bmatrix}, \qquad C = \begin{bmatrix}2 & 1\\ -3 & 0\\ 1 & 2\end{bmatrix},$$
$$D = \begin{bmatrix}-2 & 5\\ 5 & 0\end{bmatrix}, \quad E = \begin{bmatrix}-1 & 1 & 3\\ -1 & -4 & 0\\ 0 & 2 & 5\end{bmatrix}$$

6. Write the following linear function u from $\mathbb{R}^3$ to $\mathbb{R}^4$ as a matrix U. Determine if $U\vec{a}$, $U\vec{b}$, and $U\vec{c}$, where $\vec{a} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \vec{b} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \vec{c} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$, are orthogonal.

$$u(x, y, z) = \begin{bmatrix}-x + y, & x - y, & -x + z, & -y + z\end{bmatrix}$$

7. Calculate, when they are defined, the products AB and BA for all the following cases:

   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

8. Calculate AB and AC. Can A be inverted? Find all 3×3 matrices M such that $AM = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

9. Given a and b in $\mathbb{R}^*$ (i.e. real but non-zero), find all matrices B in $\mathbb{R}^2$ that commute with A, i.e. such that AB=BA, for $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

10. Calculate A^n and B^n for all n (start with $2, 3, 4 \dots$) for $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

11. Show that if $A^2 = 2I - A$, then A is invertible. Find the inverse for $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

12. Show that $A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$ is invertible and calculate its inverse.

   Hint. *If you are already familiar with Python or Matlab, you can evaluate the results of most of the exercises using code.*